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The stability of composite models

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Abstract. The stability of composite spherically symmetrical adiabatic fluid spheres consisting of a core and an envelope is examined on the basis of general relativity, the core and envelope being mixtures of ideal gas and isotropic radiation for which the ratio β of the gas pressure to the total pressure is taken to be a small (different) constant. The stability of these models is discussed on the basis of binding energy, which is considered to be a function of the total radius, R, of the configuration. It is found that the critical radius, R_c, at which instability sets in is strongly dependent on the position of the interface separating the core from the envelope and also upon the values of β in the core and envelope.

1. Introduction

Considering static spherically symmetrical fluid spheres in general relativity Tooper (1964) found that although a negative binding energy is a necessary condition for instability it is not a sufficient condition. Also Fowler (1964) argued on physical grounds that instability should set in at the first maximum of the binding energy with respect to the total radius, R, for a fixed rest mass. Using a post-Newtonian approximation to the first order in R_S/R where R_S is the Schwarzschild radius, it was found that instability sets in at a critical radius R given by

$$\frac{R_c}{R_S}=8\frac{(5-n)}{\beta}\zeta_n \tag{1.1}$$

where ζ_n is a numerical constant depending on the polytropic index n and where β is the ratio of the gas pressure to the total pressure and is assumed to be a very small constant throughout the model.

Such homogeneous models as studied here are, of course, a great oversimplification. All stars exhibit some form of composite nature, this being especially pronounced in white dwarfs and red giants. The core, which houses the internal source of nuclear energy, is represented by one set of equations whilst the envelope is characterized by another.

Also, for massive stars it has been shown by Hoyle and Fowler (1964) that provided they behave as polytropes then only when the polytropic index $n=3$ is β a small constant throughout the model being in fact given approximately by $\beta \sim (4{\cdot}3/\mu)(M_0/M)^{1/2}$. In reality β is expected to vary throughout any stellar model and increase as the surface is reached. The introduction of an envelope with larger values of β than that of the core without unduly complicating the model would thus be a desirable step forward. However, it must be emphasized that a polytropic equation of state and indeed Tooper's slightly more realistic equation of state (21), which we will use throughout this paper, are over-simplifications and are inadequate for applications to stars.

In this paper we derive an expression for the critical radius R_c of composite spherically symmetrical models for which the core and envelope are both assumed to be mixtures of ideal gas and radiation, the value of the constant β being larger in the envelope than in the core.

2. Equations of state and characteristic equations for core and envelope

2.1. Equations of state for core and envelope

The equation of state for a mixture of ideal gas and isotropic radiation for which β is a small constant is given by Tooper (1965)

$$p=K(\beta)\rho_g^{4/3}, \qquad \text{where } \rho c^2=\rho_g c^2+\frac{\beta}{\gamma-1}K(\beta)\rho_g^{4/3}+3(1-\beta)K(\beta)\rho_g^{4/3} \tag{2.1}$$

where ρ_g is the density of the rest mass of the gas, γ the ratio of specific heats of the gas and ρc^2 is the total energy-density due to all causes. The constant $K(\beta)$ is given by

$$K(\beta)=\left[\left(\frac{k}{\mu H}\right)^4\frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3} \tag{2.2}$$

where the other symbols have their conventional meanings, and the total pressure p is made up of both gas pressure and radiation pressure.

Equation (2.1) may be written in the form

$$p=K(\beta)\rho_g^{4/3} \qquad \rho c^2=\rho_g c^2+Ap \tag{2.3}$$

where

$$A=(\beta/\gamma-1)+3(1-\beta). \tag{2.4}$$

The appropriate equations of state representing core and envelope are now defined by choosing the relevant value of the constant A.

2.2. Characteristic equations for core and envelope

2.2.1. Core. Throughout this paper we used a co-moving coordinate system at rest with respect to the fluid such that the spherically symmetrical line element may be written

$$ds^2=-e^\lambda\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)+e^\nu\,dt^2.$$

In these coordinates the equations of hydrostatic equilibrium in general relativity are (Volkoff and Oppenheimer 1939)

$$\frac{1-2GM(r)/rc^2}{p+\rho c^2}r^2\frac{dp}{dr}+\frac{GM(r)}{c^2}+\frac{4\pi G}{c^4}r^3p=0 \tag{2.5}$$

$$dM(r)/dr=4\pi r^2\rho. \tag{2.6}$$

Defining σ by

$$\sigma=p_c/\rho_{gc}c^2=(K(\beta)/c^2)\rho_{gc}^{1/3} \tag{2.7}$$

where subscript c denotes central values, then defining the dimensionless variables ξ, θ and $V(\xi)$ by

$$\rho_g = \rho_{g_c}\theta^3 \tag{2.8}$$

$$r = \alpha\xi \tag{2.9}$$

$$Mr = 4\pi\rho_{g_c}\alpha^3 V(\xi) \tag{2.10}$$

where r is the distance from the centre, M_r is the mass inside radius r and

$$\alpha^2 = \sigma c^2/\pi G\rho_{g_c} \tag{2.11}$$

equations (2.5) and (2.6) become

$$\frac{1-8\sigma V(\xi)/\xi}{1+(1+A)\sigma\theta}\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + V(\xi) + \sigma\xi^3\theta^4 = 0 \tag{2.12}$$

$$\mathrm{d}V/\mathrm{d}\xi = \xi^2\theta^3(1+A\sigma\theta). \tag{2.13}$$

These are the general-relativistic equations of hydrostatic equilibrium in the core and are to be solved subject to the usual boundary conditions

$$\theta(0) = 1 \qquad V(0) = 0 \qquad (\mathrm{d}\theta/\mathrm{d}\xi \to 0 \text{ as } \xi \to 0). \tag{2.14}$$

Unlike the complete model, the total radius and mass of a model cannot, in this case, be defined since the solutions to equations (2.12) and (2.13) do not extend to the surface. We can, however, define the interfacial values of mass and radius (the interface being the point where the core joins the envelope).

The interfacial radius, r_i is given by

$$r_i = \alpha\xi_i \tag{2.15}$$

and the interfacial values of the gas density and pressure are respectively

$$\rho_{g_i} = \rho_{g_c}\theta_i^3 \tag{2.16}$$

and

$$p_i = p_c\theta_i^4 = K(\beta)\rho_{g_c}^{4/3}\theta_i^4. \tag{2.17}$$

Hence, the total energy density at the interface is given by

$$\rho_i c^2 = \rho_{g_i}c^2 + Ap_i \tag{2.18}$$

and the mass inside the interface is

$$M_i = 4\pi\rho_{g_c}\alpha^3 V(\xi_i). \tag{2.19}$$

2.2.2. *Envelope.* We shall take the equation of state within the envelope to be of the same form as (2.3) using a subscript 1 to denote values in the envelope. The equation of state then becomes

$$p_1 = K_1(\beta_1)\rho_{g_1}^{4/3} \qquad \rho_1 c^2 = \rho_{g_1} + A_1 p_1. \tag{2.20}$$

By analogy with the core we introduce the dimensionless variables ϕ and η defined by

$$\rho_{g_1} = \rho_{g_c}\phi^3 \tag{2.21}$$

where the value ρ_{g_c} is identical to that in equation (2.8) and

$$r = \alpha_1 \eta. \tag{2.22}$$

Similarly we define σ_1 and α_1 such that

$$\sigma_1 = (K_1(\beta_1)/c^2)\rho_{g_c}^{1/3} \tag{2.23}$$

$$p_1 = K_1(\beta_1)\rho_{g_c}^{4/3}\phi^4 \tag{2.24}$$

$$M_r = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta) \tag{2.25}$$

and

$$\alpha_1^2 = \sigma_1 c^2/\pi G\rho_{g_c}. \tag{2.26}$$

The equations of hydrostatic equilibrium for the envelope then take on the form

$$\frac{1-8\sigma_1 V_1(\eta)/\eta}{1+(1+A_1)\sigma_1\phi}\eta^2\frac{d\phi}{d\eta} + V_1(\eta) + \sigma_1\eta^3\phi^4 = 0 \tag{2.27}$$

$$\frac{dV_1}{d\eta} = \eta^2\phi^3(1 + A_1\sigma_1\phi). \tag{2.28}$$

Although in general the solutions to (2.27) and (2.28) will not be the usual general-relativistic generalizations of the Lane–Emden solutions since they do not extend to the centre, we can, nevertheless (Hargreaves 1972), define the total mass, radius etc of the model. The outer surface is taken to correspond to the smallest positive value η_s for which

$$\phi(\eta_s) = 0.$$

The total radius and the total mass are then given by

$$R = \alpha_1\eta_s \tag{2.29}$$

and

$$M = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta_s). \tag{2.30}$$

Similarly, the interfacial values of the various parameters within the envelope may be written

$$r_i = \alpha_1\eta_i \tag{2.31}$$

$$p_{1_i} = K_1(\beta_1)\rho_{g_c}^{4/3}\phi_i^4 \tag{2.32}$$

$$\rho_{g1_i}c^2 = \rho_{g_c}\phi_i^3 \tag{2.33}$$

and

$$\rho_{1_i} = \rho_{g1_i}c^2 + A_1 p_{1_i} \tag{2.34}$$

and the mass inside r_i given by

$$M_i = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta_i). \tag{2.35}$$

3. Interfacial boundary conditions

The equations of equilibrium, both in the core and in the envelope, are solved subject to the boundary conditions at the interface.

Since pressure is to be continuous everywhere, including the interface, the values assigned to this quantity in equations (2.17) and (2.32) must be identical. Thus

$$p_i = p_{1_i} = K(\beta)\rho_{gi}^{4/3} = K_1(\beta_1)\rho_{g1_i}^{4/3}. \tag{3.1}$$

Using the definitions of σ and σ_1 we have

$$\sigma_1/\sigma = K_1(\beta_1)/K(\beta) \tag{3.2}$$

so that on using (3.2), (3.1) and the definitions of θ and ϕ we obtain

$$\sigma_1/\sigma = (\theta_i/\phi_i)^4. \tag{3.3}$$

We can obtain further relations from the fact that the respective values of r_i and M_i in core and envelope must be identical. Hence

$$r_i = \alpha\xi_i = \alpha_1\eta_i \tag{3.4}$$

and

$$M_i = 4\pi\rho_{gc}\alpha^3 V(\xi_i) = 4\pi\rho_{gc}\alpha_1^3 V_1(\eta_i). \tag{3.5}$$

This gives

$$\alpha^3 V(\xi_i) = \alpha_1^3 V_1(\eta_i) \tag{3.6}$$

or

$$\eta_i = (\alpha/\alpha_1)\xi_i = (\sigma/\sigma_1)^{1/2}\xi_i \tag{3.7}$$

and

$$V_1(\eta_i) = (\alpha/\alpha_1)^3 V(\xi_i) = (\sigma/\sigma_1)^{3/2} V(\xi_i). \tag{3.8}$$

One further condition is needed in order to solve the equations of equilibrium and this is afforded by the continuity of temperature. This is chosen since it has been shown by Durgapal and Gehlot (1971) that density may have discontinuity across the boundary.

Since we are considering a perfect gas

$$p = \frac{1}{\beta}\left(\frac{k}{\mu H}\right)\rho_g T \tag{3.9}$$

thus, considering interfacial values of pressure we obtain

$$(1/\beta)(k/\mu H)\rho_{gc}\theta_i^3 T_i = (1/\beta_1)(k/\mu_1 H)\rho_{gc}\phi_i^3 T_i \tag{3.10}$$

where μ and μ_1 are the values of the mean molecular weights of the gases in the core and envelope respectively.

Rewriting we have

$$(-\beta/\beta_1)(\mu/\mu_1) = (\theta_i/\phi_i)^3 = (\sigma_1/\sigma)^{3/4}. \tag{3.11}$$

The above boundary conditions (3.3), (3.7), (3.8) and (3.11) are sufficient to solve the equations (2.12), (2.13), (2.27) and (2.28).

4. Binding energy

We define the binding energy E_b of a star as the total energy of the bound system exclusive of the rest mass energy of the unbound particles dispersed to infinity at zero temperature. Hence

$$E_b = (M_{0g} - M)c^2. \tag{4.1}$$

In this expression, M_{0g} is the rest mass of the gas and M is the total mass of the star.

For our composite model we can write (4.1) as

$$E_b = \int_0^{\xi_i} 4\pi\rho_g c^2 e^{\lambda/2} r^2\, dr + \int_{\eta_i}^{\eta_s} 4\pi\rho_g c^2 e^{\lambda/2} r^2\, dr - \int_0^{\xi_i} 4\pi\rho c^2 r^2\, dr - \int_{\eta_i}^{\eta_s} 4\pi\rho c^2 r^2\, dr \tag{4.2}$$

where the 'proper' element of volume has been employed to calculate the rest mass M_{0g}. Substituting for $e^{\lambda/2}$ and eliminating ρ_g we can write

$$\begin{aligned} -E_b = &\int_0^{\xi_i} 4\pi A p[1-(2GM_r/rc^2)]^{-1/2} r^2\, dr + \int_{\eta_i}^{\eta_s} 4\pi A_1 p_1 [1-(2GM_r/rc^2)] r^2\, dr \\ &+ \int_0^{\xi_i} 4\pi\rho c^2\{1-[1-(2GM_r/rc^2)]^{-1/2}\} r^2\, dr \\ &+ \int_{\eta_i}^{\eta_s} 4\pi\rho_1 c^2\{1-[1-(2GM_r/rc^2)]^{-1/2}\} r^2\, dr \end{aligned} \tag{4.3}$$

where any dynamical energy arising from bulk motions throughout the star has been neglected since we are considering the state of hydrostatic equilibrium.

In terms of the dimensionless variables ξ, θ, V, η, ϕ, V_1, equation (4.3) becomes

$$\begin{aligned} -E_b = &\left(\frac{A}{3}-1\right)\left[\frac{GM^2\eta_s}{RV_1(\eta_s)^2}\left(\frac{\sigma}{\sigma_1}\right)^{5/2}\int_0^{\xi_i} V\xi\theta^3\, d\xi + \frac{A}{4}\frac{G^2M^3\eta_s^2}{R^2c^2V_1(\eta_s)^3}\left(\frac{\sigma}{\sigma_1}\right)^{7/2}\int_0^{\xi_i} V\xi\theta^4\, d\xi\right] \\ &+\left(\frac{A_1}{3}-1\right)\left(\frac{GM^2\eta_s}{RV_1(\eta_s)^2}\int_{\eta_i}^{\eta_s} V_1\eta\phi^3\, d\eta + \frac{A_1}{4}\frac{G^2M^3\eta_s^2}{R^2c^2V_1(\eta_s)^3}\int_{\eta_i}^{\eta_s} V_1\eta\phi^4\, d\eta\right) \\ &+\frac{A}{6}\left(\frac{\sigma}{\sigma_1}\right)^{7/2}\frac{G^2M^3\eta_s^2}{R^2c^2V_1(\eta_s)^3}\int_0^{\xi_i} V\xi\theta^4\, d\xi + \frac{A_1}{6}\frac{G^2M^3\eta_1^2}{R^2c^2V_1(\eta_s)^3}\int_{\eta_i}^{\eta_s} V_1\eta\phi^4\, d\eta \\ &+\left(\frac{A}{3}-\frac{1}{2}\right)\left[\left(\frac{\sigma}{\sigma_1}\right)^{7/2}\frac{3G^2M^3\eta_s^2}{R^2c^2V_1(\eta_s)^3}\int_0^{\xi_i} V^2\theta^3\, d\xi\right. \\ &\left.+\left(\frac{\sigma}{\sigma_1}\right)^{9/2}\frac{3A}{4}\frac{G^3M^4\eta_s^3}{R^3c^4V_1(\eta_s)^4}\int_0^{\xi_i} V^2\theta^4\, d\xi\right] \\ &+\left(\frac{A_1}{3}-\frac{1}{2}\right)\left(\frac{3G^2M^3\eta_s^2}{R^2c^2V_1(\eta_s)^3}\int_{\eta_i}^{\eta_s} V_1^2\phi^3\, d\eta\right. \\ &\left.+\frac{3A_1}{4}\frac{G^3M^4\eta_s^3}{R^3c^4V_1(\eta_s)^4}\int_{\eta_i}^{\eta_s} V_1^2\phi^4\, d\eta\right) \end{aligned} \tag{4.4}$$

where we have used

$$R_S/R = 2GM/Rc^2 = 8\sigma_1 V_1(\eta_s)/\eta_s. \tag{4.5}$$

Collecting terms and neglecting those of the order $(2GM/Rc^2)^3$ and above we obtain in the post-Newtonian approximation

$$\begin{aligned}\frac{E_b}{Mc^2}=&\frac{1}{2}\left(\frac{A}{3}-1\right)\left(\frac{\sigma}{\sigma_1}\right)^{5/2}\frac{\eta_s}{V_1(\eta_s)}\left(\frac{R_S}{R}\right)\int_0^{\xi_i}V\xi\theta^3\,d\xi\\&+\frac{1}{2}\left(\frac{A_1}{3}-1\right)\frac{\eta_s}{V_1(\eta_s)}\left(\frac{R_S}{R}\right)\int_{\eta_i}^{\eta_s}V_1\eta\phi^3\,d\eta\\&+\frac{A}{48}(A-1)\left(\frac{\sigma}{\sigma_1}\right)^{7/2}\frac{\eta_s^2}{V_1(\eta_s)^3}\left(\frac{R_S}{R}\right)^2\int_0^{\xi_i}\xi V\theta^4\,d\xi\\&+\frac{A_1}{48}(A_1-1)\frac{\eta_s^2}{V_1(\eta_s)^3}\left(\frac{R_S}{R}\right)^2\int_{\eta_i}^{\eta_s}V_1\eta\phi^4\,d\eta\\&+\frac{3}{4}\left(\frac{A}{3}-\frac{1}{2}\right)\left(\frac{\sigma}{\sigma_1}\right)^{7/2}\frac{\eta_s^2}{V_1(\eta_1)^3}\left(\frac{R_S}{R}\right)^2\int_0^{\xi_i}V^2\theta^3\,d\xi\\&+\frac{3}{4}\left(\frac{A_1}{3}-\frac{1}{2}\right)\frac{\eta_s^2}{V_1(\eta_s)^3}\left(\frac{R_S}{R}\right)^2\int_{\eta_i}^{\eta_s}V_1^2\,\phi^3\,d\eta.\end{aligned}\tag{4.6}$$

This may be written

$$\frac{E_b}{Mc^2}=X\left(\frac{R_S}{R}\right)-Y\left(\frac{R_S}{R}\right)^2\tag{4.7}$$

where

$$X=\frac{\eta_s}{2V_1(\eta_s)}\left[\left(\frac{A}{3}-1\right)\left(\frac{\sigma}{\sigma_1}\right)^{5/2}\int_0^{\xi_i}V\xi\theta^3\,d\xi+\left(\frac{A_1}{3}-1\right)\int_{\eta_i}^{\eta_s}V_1\eta\phi^3\,d\eta\right]\tag{4.8}$$

and

$$\begin{aligned}Y=-\frac{\eta_s^2}{V_1(\eta_s)^3}\Bigg[&\left(\frac{\sigma}{\sigma_1}\right)^{7/2}\frac{A}{48}(A-1)\int_0^{\xi_i}\xi V\theta^4\,d\xi+\frac{A_1}{48}(A_1-1)\int_{\eta_i}^{\eta_s}\eta V\phi^4\,d\eta\\&+\frac{3}{4}\left(\frac{A}{3}-\frac{1}{2}\right)\left(\frac{\sigma}{\sigma_1}\right)^{7/2}\int_0^{\xi_i}V^2\theta^3\,d\xi+\frac{3}{4}\left(\frac{A_1}{3}-\frac{1}{2}\right)\int_{\eta_i}^{\eta_s}V^2\phi^3\,d\eta\Bigg].\end{aligned}\tag{4.9}$$

It can be shown that (4.6) reduces exactly to the expression for binding energy obtained by Fowler (1964) for a complete polytropic model of index three providing we extend the interface to the surface and make the same assumptions as Fowler that $A \simeq 3$ except in the classical term where $(\frac{1}{3}A-1)=\frac{1}{2}$. Then from figure 1 which shows the dimensionless binding energy as a function of the relativistic parameter σ for various values of β, assuming that the interface is extended to the surface so that the model consists of all core, it can be seen that, except perhaps for $\sigma = 0{\cdot}005$ and $\beta = 0{\cdot}1$, these models would be unstable for all values of β and σ which we are considering. Thus anticipating the results exhibited in figures 2 and 3, for models consisting of core and envelope for various positions of the interface ξ_i, we see that the introduction of an envelope exerts a stabilizing influence. Further, from figure 3 we see that for small ξ_i (model consisting of nearly all envelope) the critical radius is comparatively small so that models consisting of all envelope are very stable.

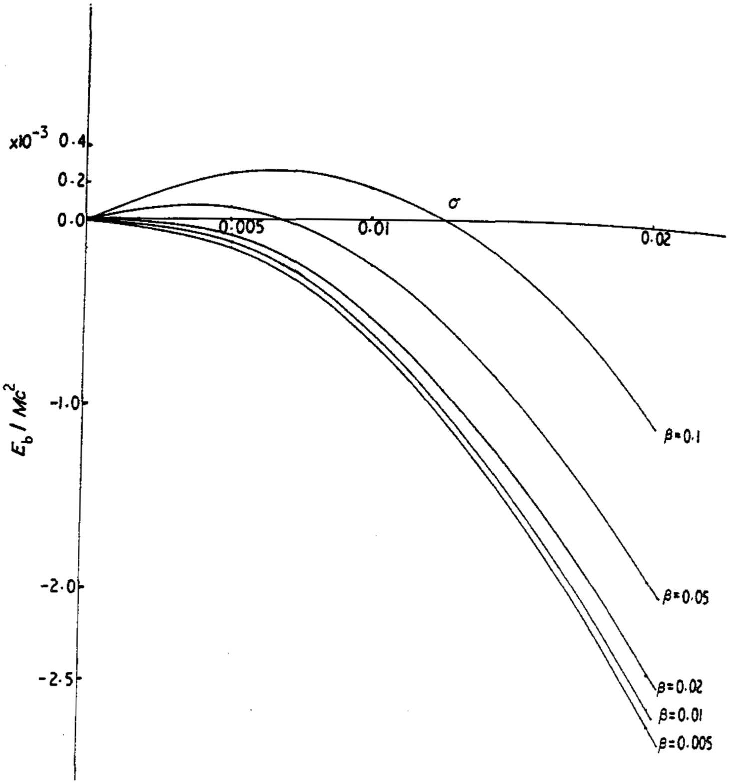

Figure 1. Dimensionless binding energy against σ for configurations consisting of all core.

5. Critical radius

Using the expression (4.7) for the binding energy of the model we obtain the value of the critical radius, R_c by differentiating with respect to the total radius, and equating to zero:

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{E_b}{Mc^2}\right) = -X\left(\frac{R_S}{R^2}\right) + 2Y\left(\frac{R_S^2}{R^3}\right). \tag{5.1}$$

Thus

$$R_S/R_c = X/2Y \tag{5.2}$$

where X and Y are given by (4.8) and (4.9). A summary of values of (R_c/R_S) for various models is given in table 1. It is seen that (R_c/R_S) depends strongly on the position of the interface as well as on the values of β and σ, and β_1 in the envelope.

6. Numerical results

The equations of hydrostatic equilibrium, in the core ((2.12) and (2.13)) and in the envelope ((2.27) and (2.28)) were solved numerically using a fourth-order Runge-

Table 1. Summary of values of R_c/R_S and $R_cK_3(\rho_{g_c})\times10^2$ for various values of ξ_i, σ, β and β_1.

ξ_i	σ	β	β_1	$R_cK_3(\rho_{g_c})\times10^2$	R_c/R_S
0·5	0·005	0·005	0·05	$6\cdot52\times10^{-1}$	72·03
1·0				1·65	71·30
1·5				∞	∞
0·5	0·005	0·005	0·1	$1\cdot94\times10^{-1}$	27·18
1·0				$6\cdot60\times10^{-1}$	30·93
1·5				1·58	43·20
2·0				3·30	67·34
2·5				∞	∞
0·5	0·005	0·01	0·05	1·58	101·27
1·0				2·49	87·69
1·5				4·23	101·07
2·0				∞	∞
0·5	0·005	0·01	0·1	$3\cdot09\times10^{-1}$	34·13
1·0				$7\cdot90\times10^{-1}$	34·07
1·5				1·69	44·52
2·0				3·28	65·64
2·5				∞	∞
0·5	0·005	0·02	0·05	5·71	128·00
1·0				6·06	123·18
1·5				7·20	130·05
2·0				∞	∞
0·5	0·005	0·02	0·1	$7\cdot49\times10^{-1}$	47·88
1·0				1·18	41·73
1·5				2·03	48·57
2·0				3·41	65·01
2·5				5·42	91·44
3·0				∞	∞
0·5	0·01	0·005	0·1	$5\cdot45\times10^{-1}$	27·30
1·0				1·86	31·46
1·5				∞	∞
0·5	0·01	0·01	0·1	$8\cdot64\times10^{-1}$	34·27
1·0				2·21	34·61
1·5				∞	∞
0·5	0·01	0·01	0·5	$4\cdot98\times10^{-2}$	2·82
1·0				$2\cdot26\times10^{-1}$	4·01
1·5				$6\cdot64\times10^{-1}$	6·75
2·0				1·62	12·32
2·5				3·60	23·43
3·0				7·61	45·59
3·5				∞	∞
0·5	0·01	0·02	0·1	2·08	48·25
1·0				3·30	42·34
1·5				5·70	49·97
2·0				∞	∞
0·5	0·01	0·02	0·5	$5\cdot88\times10^{-2}$	3·07
1·0				$2\cdot39\times10^{-1}$	4·10
1·5				$6\cdot70\times10^{-1}$	6·72
2·0				1·59	11·98
2·5				3·40	22·05
3·0				6·80	40·66
3·5				∞	∞

Table 1—continued.

ξ_i	σ	β	β_1	$R_cK_3(\rho_{gc})\times 10^2$	R_c/R_S
0·5	0·01	0·05	0·5	$9\cdot89\times 10^{-2}$	3·922
1·0				$2\cdot83\times 10^{-1}$	4·426
1·5				$6\cdot98\times 10^{-1}$	6·697
2·0				1·51	11·17
2·5				2·96	18·94
3·0				5·23	31·09
3·5				8·34	47·68
4·0				∞	∞
0·5	0·01	0·1	0·5	$2\cdot32\times 10^{-1}$	5·37
1·0				$4\cdot00\times 10^{-1}$	5·13
1·5				$7\cdot84\times 10^{-1}$	6·87
2·0				1·47	10·41
2·5				2·55	15·93
3·0				3·96	23·23
3·5				5·52	31·38
4·0				6·99	39·13
0·1	0·02	0·005	0·05	2·85	124·62
0·3				3·42	87·92
0·5				∞	∞
0·1	0·02	0·005	0·1	$3\cdot61\times 10^{-1}$	48·05
0·3				$7\cdot59\times 10^{-1}$	30·13
0·5				1·52	27·54
0·7				2·65	28·29
1·0				∞	∞
0·1	0·02	0·01	0·1	1·34	58·83
0·3				1·62	41·58
0·5				2·39	34·55
0·7				3·57	33·14
1·0				∞	∞
0·1	0·02	0·02	0·1	5·24	62·64
0·3				5·27	57·00
0·5				5·68	48·99
0·7				6·64	44·43
1·0				∞	∞

Kutta method. The integrations in equation (4.6) were then evaluated in order to obtain a numerical result for the binding energy and the ratio of (R_c/R_S). A summary of the results for various σ, β and β_1, is given in table 1.

Values of R_c/R_S were obtained by applying (5.2) to all those models with a positive binding energy. Those which exhibited a negative binding energy were taken to represent unbound systems having an infinite critical radius. A summary of results of the ratio (R_c/R_S) against the interfacial radius ξ_i is shown in figure 2 for various β and β_1, and $\sigma = 0\cdot005$.

The interesting fact is that these graphs exhibit a minimum. This can be explained because of the rapid increase in mass of these models with increasing ξ_i. Since R_S is directly proportional to the total mass M, of the model, as the mass increases so does R_S. Thus, although the critical radius R_c, may increase with ξ_i for one particular model, this

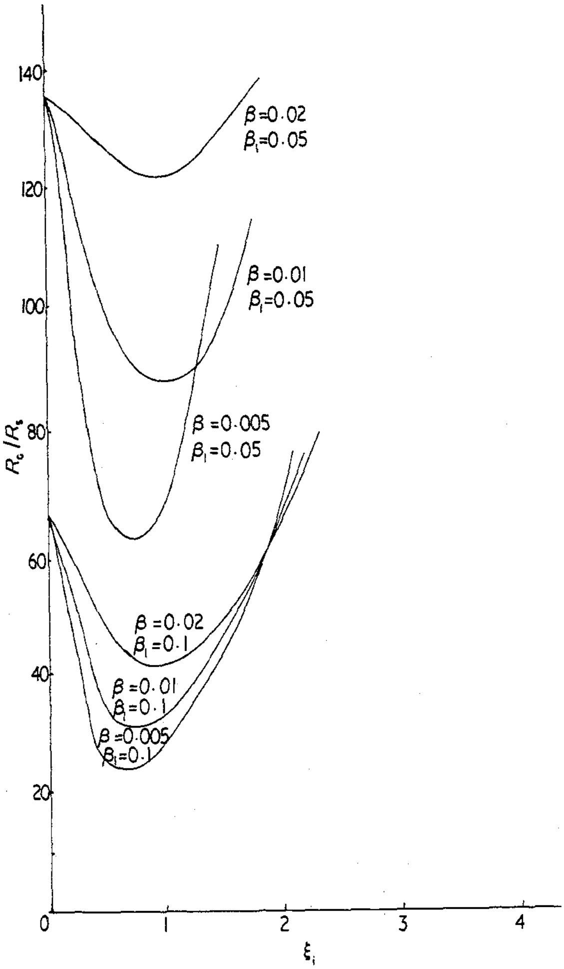

Figure 2. R_c/R_S against ξ_i for $\sigma = 0{\cdot}005$.

increase may be completely masked by the increase in R_S. We shall thus consider the critical radius.

To obtain graphs showing the way in which the critical radius varies with ξ_i we make use of the values of $V_1(\eta_s)$ obtained at each ξ_i from the solutions of equations (2.12) and (2.13) and (2.27) and (2.28).

Consider

$$M = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta_s) \tag{6.1}$$

and so substituting for α_1 we have

$$M = \frac{4\sigma_1^{3/2}c^3}{(\pi\rho_{g_c})^{1/2}G^{3/2}} V_1(\eta_s). \tag{6.2}$$

The graphs of (R_c/R_S) against ξ_i give us the value of $K_1(R_c/M)$ at various ξ_i, where $K_1 = c^2/2G$. Thus multiplying the value of (R_c/R_S) at ξ_i (call it $(R_c/R_S)_{\xi_i}$) and the value of $V_1(\eta_s)$ for this model (call this $[V_1(\eta_s)]_{\xi_i}$) we obtain:

$$\left(\frac{R_c}{R_S}\right)_{\xi_i}[V_1(\eta_s)]_{\xi_i}=K_1K_2(\rho_{g_c})\left(\frac{1}{\sigma}\right)^{3/2}(R_c)_{\xi_i} \tag{6.3}$$

where $K_2(\rho_{g_c})=(\pi G^3\rho_{g_c})^{1/2}/4c^3$ is a function of the central density, thus

$$\left(\frac{R_c}{R_S}\right)_{\xi_i}[V_1(\eta_s)]_{\xi_i}=K_3(\rho_{g_c})(R_c)_{\xi_i} \tag{6.4}$$

where

$$K_3(\rho_{g_c})=K_1K_2(\rho_{g_c})=(\pi G\rho_{g_c})^{1/2}/8c^2.$$

Hence, by multiplying our two graphs together and multiplying by $\sigma_1^{3/2}$ (where σ_1 is the value of the relativity parameter in the envelope for the particular model in question) we can plot a graph of the same form as R_c against ξ_i for a model with fixed ρ_{g_c}. We multiply by $\sigma_1^{3/2}$ even though σ_1 is a constant throughout one model so as to maintain the correct relationship between different graphs with varying σ_1. Differing central densities between models are taken acount of by altering the value of $K_3(\rho_{g_c})$ which then has the effect of moving the graph up or down relative to the axis $R_cK_3(\rho_{g_c})$.

Plots of $R_cK_3(\rho_{g_c})$ against ξ_i are shown in figure 3 where a constant central density has been assumed throughout for all models. It is seen that the critical radius depends strongly on the position of the interface, increasing steadily for small values of ξ_i and then more rapidly as ξ_i increases, the core having more and more effect. Initially (when the configuration consists mainly of envelope) the value of σ_1 for a given small β_1, determines the relative stability of the models, but as ξ_i increases (the configuration consisting of more and more core) the value of R_c becomes strongly dependent on the value of β in the core. The effect of σ and β_1 on the stability of the models is also clearly demonstrated. Models with σ equal to 0·3 and above were found to have no stable solutions for $\xi_i \geqslant 0{\cdot}5$, whereas those models with σ equal to 0·2 and below exhibited stable solutions, stability increasing with decreasing σ.

On the other hand those models with larger values of β_1 in the envelope are seen to be far more stable than those with smaller values of β_1 as expected.

Also as the core increases in size the models become more and more unstable, nearly all becoming unbound before reaching the state where they consist entirely of core. It is interesting, therefore, to consider under what conditions a model will remain stable even when the stabilizing effect of the envelope has been removed. For small σ and β we can make use of Fowler's approximation for the binding energy of a complete polytropic model of index 3:

$$-E_b/Mc^2=-\tfrac{3}{8}\beta(R_S/R)+\zeta_3(R_S/R)^2+\dots \tag{6.5}$$

where

$$\zeta_3=\tfrac{3}{16}(3/\pi)^{1/2}R_3. \tag{6.6}$$

For the stability of this model

$$\mathrm{d}/\mathrm{d}R(E_b/Mc^2)\geqslant 0$$

so that the condition for stability is

$$\beta\geqslant\tfrac{16}{3}\zeta_3(R_S/R). \tag{6.7}$$

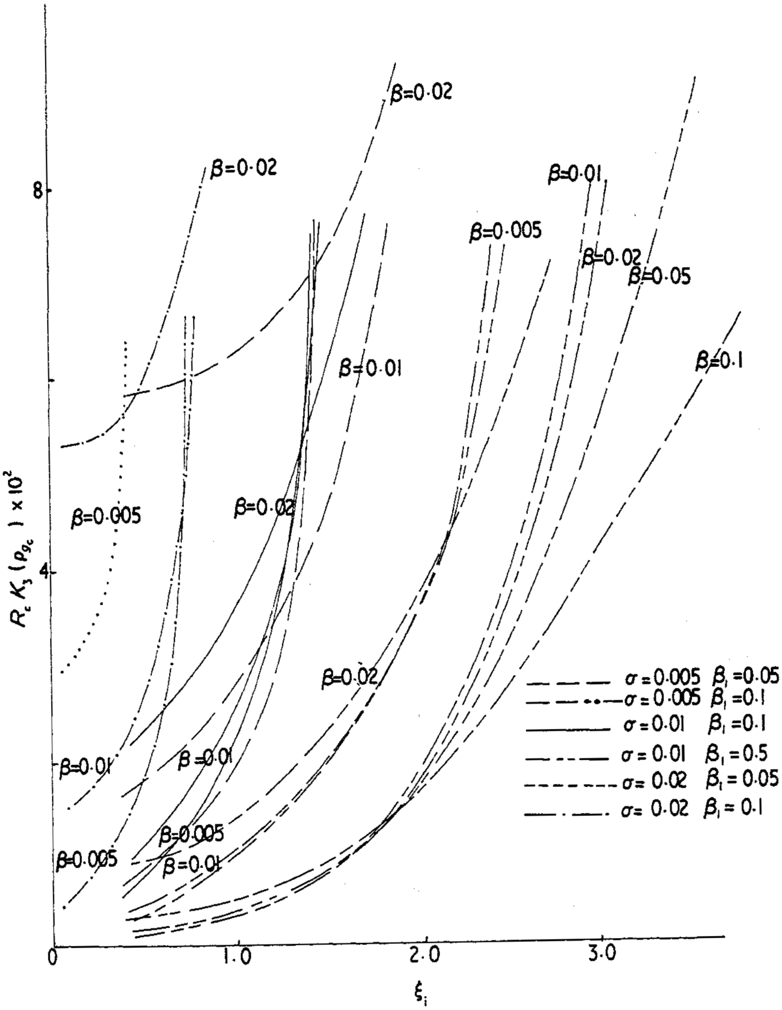

Figure 3. $R_c K_3(\rho_{g_c})$ against ξ_i.

Substituting for (R_S/R) from (4.5) and using (6.6) we obtain

$$\beta \geqslant (3/\pi)^{1/2}(8\sigma_1 V_1(\eta_s)/\eta_s). \tag{6.8}$$

It should be remembered that (6.8) is only an approximation for the models in this paper giving us an idea of the size of β in the core needed for the model to remain stable for all ξ_i. And as expected the only model to remain stable for the complete range of ξ_i used in this paper is the only one to satisfy this inequality.

In conclusion we see that given a core consisting of matter and radiation for which β, the ratio of gas pressure to total pressure, is a small constant, the fitting of an envelope onto this core has a significant effect on the stability of the configuration. The critical radius at which instability sets in decreases rapidly as the size of the envelope relative to the core increases. Also the stabilizing effect of the envelope increases with increasing β_1.

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